LECTURE 22 LINEARISATION AND DIFFERENTIALS

Linearisation continued

Last class, we defined that the linearisation $L(x)$ of a function $f(x)$ at a particular point $x = a$ is the tangent line to $f(x)$ at this point. More precisely, given $(a, f(a))$ a point on the graph of $f(x)$, the linearisation satisfies

$$L(x) = f(a) + f'(a)(x - a).$$

The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at $x = a$. The point $x = a$ is the **center** of the approximation.

Example. Find the linearsation of $f(x) = \sqrt{x+1}$ at $x = 0$.

Solution.

$$L(x) = f(0) + f'(0)(x - 0) = 1 + \frac{1}{2\sqrt{0+1}}(x - 0) = 1 + \frac{1}{2}x.$$

Let's also examine how accurate this approximation $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ is.

Approximation	True Value	$\lvert$True Value − Approximation$\rvert$
$x = 0.005,\quad \sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.0025$	1.002497	$0.000003 < 10^{-5}$
$x = 0.01,\quad \sqrt{1.01} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$x = 0.2,\quad \sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$x = 3,\quad \sqrt{1+3} \approx 1 + \frac{3}{2} = 2.5$	2	0.5

So, we see that near $x = 0$, this approximation is not terrible. But as we venture away from $x = 0$, say to $x = 3$, the error is relatively large. We then must consider linearisation near $x = 3$ to have a better estimate.

Example. We then continue to find the linearisation of $f(x) = \sqrt{x+1}$ at $x = 3$.

$$L(x) = f(3) + f'(3)(x - 3) = 2 + \frac{1}{2\sqrt{3+1}}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

Then, let's check how good this approximation is, near $x = 3$. Consider $x = 3.2$. Then, the linearsation says

$$\sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 2.05$$

where the true value is

$$\sqrt{1+3.2} \approx 2.04939.$$

However, with the linearisation given in the previous example,

$$\sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 2.6,$$

which is obviously way off.

Example. Linearisation of $f(x) = (1+x)^k$ at $x = 0$.

Solution. The linearisation at $x = 0$ is

$$f(x) \approx L(x) = f(0) + f'(0)x = 1 + k(1+0)^{k-1}x = 1 + kx.$$

This approximation works for any real number k near $x = 0$. That is,

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x;$$
$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x;$$
$$\sqrt[3]{1+5x^4} = \left(1+5x^4\right)^{\frac{1}{3}} \approx 1 + \frac{1}{3}\left(5x^4\right) = 1 + \frac{5}{3}x^4;$$
$$\frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}} \approx 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) = 1 + \frac{1}{2}x^2.$$

Differentials

Definition. Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy (**a dependent variable**) is

$$dy = f'(x)\,dx.$$

Note that dy always depends on x **AND** dx.

One of the goals of differentials is to make estimate of things that are hard to compute directly. For example, we know $\sqrt{4}$ very well. Can we use this to estimate $\sqrt{4.02}$?

Example. Find dy if $y = x^5 + 37x$. Find its value when $x = 1$ and $dx = 0.2$.

Solution. $dy = \left(5x^4 + 37\right)dx$. And thus

$$dy = \left(5(1)^4 + 37\right) \cdot 0.2 = 8.4.$$

The differential dx is a deep concept. It shows **infinitesimal change** in x-values – a very very small change, almost negligible. But when paired with the instantaneous rate of change, i.e. the derivative, it produces also an infinitesimal change in y. dx is different from Δx, with the former being a sizeable change in x-values, which also leads to a sizeable change in y-values, that is, Δy.

$$\Delta y = f(x + \Delta x) - f(x).$$

Below, we discuss the relationship between $\Delta x, dx, \Delta y, dy$, and how it can be represented by ideas from linearisation.

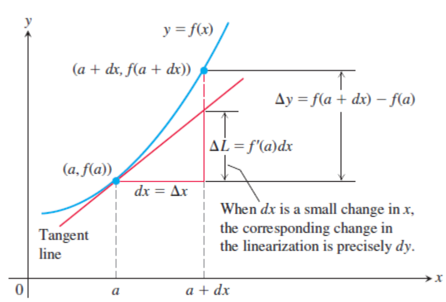

FIGURE 3.44 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$is

$$\Delta y = f(a + dx) - f(a) = f(a + \Delta x) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= f(a) + f'(a)\left[(a + dx) - a\right] - f(a)\\ &= f'(a)\,dx.\end{aligned}$$

That is, the change in the linearisation of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$. That is, $dy = \Delta L$, when $dx = \Delta x$.

In other words, when $dx = \Delta x$,

$$f(a + dx) \approx f(a) + \Delta L = f(a) + dy = f(a) + f'(a)\, dx,$$

that is, we can approximate a function value $f(a + dx)$ with the knowledge of $f(a)$, $f'(a)$ and the horizontal distance dx from the point of estimate $a + dx$ to the known point a.

Every differential formula such as the sum rule,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

or

$$\frac{d}{dx}(\sin(u)) = \cos(u)\frac{du}{dx}$$

has a differential form,

$$d(u + v) = du + dv, \quad \text{or } d(\sin(u)) = \cos(u)\, du$$

(as if you can cancel the dx from both sides, only when $dx \neq 0$).

Example. Find the differential of the following functions.

$$d(\tan(2x)) = \sec^2(2x)\, d(2x) = 2\sec^2(2x)\, dx;$$

$$d\left(\frac{x}{x+1}\right) = \frac{(x+1)\, dx - x d(x+1)}{(x+1)^2} = \frac{xdx + dx - xdx}{(x+1)^2} = \frac{dx}{(x+1)^2}.$$

We can also make nice estimate about physical quantities when direct measurement is difficult.

Example. The radius r of a circle increases from $a = 10$ m to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution. Since $A = \pi r^2$, the estimated increase (about $x = a$) is

$$dA = A'(a)\, dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi\, \text{m}^2.$$

Therefore, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$A(10 + 0.1) \approx A(10) + 2\pi = \pi(10)^2 + 2\pi = 102\pi,$$

i.e. the area of a circle with radius 10.1 m is approximately $102\pi\, \text{m}^2$.

However, we do know the true area:

$$A(10.1) = \pi(10.1)^2 = 102.01\pi\, \text{m}^2.$$

So, with our differential estimate, we incur an error of $0.01\pi\, \text{m}^2$, which is the difference $\Delta A - dA$ where $\Delta A = A(10 + 0.1) - A(10.1)$.

Example. Use differentials to estimate $(7.97)^{1/3}$.

Solution. We note that the function we need to deal with is $f(x) = x^{1/3}$, and we are centering about $a = 8$. Note this a is of your freedom, and we choose $a = 8$ because it is easy to evaluate $f(8)$ (you are welcome to choose $a = 7$, but it's hard to compute anyways, while incurring a lot of error).

Now, with $a = 8$, let's re-express 7.97 as

$$a + dx = 7.97 \implies dx = -0.03.$$

Using the differential for $f(x) = x^{1/3}$, we find

$$df = f'(x)\, dx = \frac{1}{3x^{2/3}} dx$$

we have

$$\begin{aligned} f(7.97) = f(a + dx) &\approx f(a) + df \\ &= 8^{1/3} + \frac{1}{3(8)^{2/3}}(-0.03) \\ &= 2 + \frac{1}{12}(-0.03) \\ &= 1.9975. \end{aligned}$$

The true value of $(7.97)^{1/3}$ is 1.997497, so we are accurate up to 6 decimals, which is pretty impressive.

References

Thomas, Calculus, 14th Edition.